

LINEAR VECTOR SPACES (L.V.S)- Algebraic Foundations

First, let's define a ket:

$$|\beta\rangle = \text{ket} \rightarrow \text{Actually, this is a column vector}$$

(Here, β is a symbol (label), the ket itself represents a column vector.)

The ket-bra notation is a commonly used notation in linear algebra and quantum mechanics.

$$|\beta\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{2 \times 1}$$

Similarly, we define the concept of a **bracket** and **bra-ket** notation:

$$\text{Bra} : \langle 1|$$

$$\text{Bra-ket} : \langle 1|1\rangle$$

A set S consisting of elements:

$$S = \{|\alpha\rangle, |\beta\rangle, \dots, |\omega\rangle\}$$

is called a **linear vector space** if it satisfies the following conditions. First let us see first the two Operations Defined in This Space:

Within this space, we define two fundamental operations:

1. **Scalar Multiplication:** Multiplication of a ket by a scalar:

$$c|\alpha\rangle$$

2. **Addition of Two Kets:** The sum of two kets:

$$|\alpha\rangle + |\beta\rangle$$

Conditions for a Linear Vector Space (LVS)

The space S is considered a **linear vector space** (LVS) if it satisfies the following properties:

1. **Closure property I: under Scalar Multiplication:** If $|\alpha\rangle \in S$, then for any scalar c :

$$c|\alpha\rangle \in S$$

2. **Closure property II: under Addition:** If $|\alpha\rangle \in S$ and $|\beta\rangle \in S$, then their sum must also belong to S :

$$|\alpha\rangle + |\beta\rangle \in S$$

3. **Existence of an Identity Element (for Addition):** There exists a unique zero ket $|0\rangle \in S$ such that:

$$|\alpha\rangle + |0\rangle = |\alpha\rangle$$

Here, $|0\rangle$ is called the *null ket* or *zero ket*.

4. **Inverse Element (for Addition)**

The inverse element in addition satisfies:

$$-|\alpha\rangle \in S, \quad |\alpha\rangle + (-|\alpha\rangle) = |0\rangle$$

★ If a space satisfies these four properties (closure under scalar multiplication and addition, existence of a zero element, and existence of an inverse element), then it is a **linear vector space**.

2) Linear Independence and the Concept of Dimension

A set of vectors $\{|n\rangle\}$ is linearly independent if the equation

$$\sum_{n=1}^{\omega} c_n |n\rangle = |0\rangle$$

holds only when all coefficients are zero:

$$c_1 |\alpha\rangle + c_2 |\beta\rangle + \cdots + c_{\omega} |\omega\rangle = |0\rangle$$

$$\Rightarrow c_1 = c_2 = \cdots = c_{\omega} = 0$$

In this case, the set $\{|\alpha\rangle, |\beta\rangle, \dots, |\omega\rangle\}$ is **linearly independent**.

Example 1

Consider the kets:

$$|\alpha\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\beta\rangle = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

Checking linear independence:

$$c_1 |\alpha\rangle + c_2 |\beta\rangle = |0\rangle$$

$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From the system of equations:

$$c_1 = 0, \quad c_2 = 0$$

Since the only solution is $c_1 = 0$ and $c_2 = 0$, the vectors $|\alpha\rangle$ and $|\beta\rangle$ are linearly independent.

Example 2

Consider another set:

$$|\alpha\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\beta\rangle = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$c_1 |\alpha\rangle + c_2 |\beta\rangle = \begin{pmatrix} c_1 + 2c_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From the equations:

$$c_1 + 2c_2 = 0,$$

Since there are infinitely many solutions for non-zero c_1 and c_2 second equation implies $c_2 = 0$, we substitute into the first equation:

$$c_1 + 2(0) = 0 \Rightarrow c_1 = 0$$

Since the only solution is $c_1 = 0$ and $c_2 = 0$, the vectors $|\alpha\rangle$ and $|\beta\rangle$ are linearly dependent.

$$|\beta\rangle = 2|\alpha\rangle$$

Dimension of a Linear Vector Space

The dimension of a linear vector space is the maximum number of linearly independent kets it contains.

For example, in the previous two examples:

- in the 2.example, Since there is only one linearly independent ket, the dimension of the space is 1.
- in the 1.example, Since there are two linearly independent kets, the dimension of the space is 2.

3) Scalar Product and Norm

Scalar Product

$$(|\alpha\rangle, |\beta\rangle) = c$$

where c is a scalar.

Here:

- The first factor $|\alpha\rangle$ is the first vector.
- The second factor $|\beta\rangle$ is the second vector.

If

$$c = 0$$

then:

$$(|\alpha\rangle, |\beta\rangle) = 0$$

which means that $|\alpha\rangle$ and $|\beta\rangle$ are **orthogonal** (perpendicular to each other).

Norm

The norm of a vector is found by taking its scalar product with itself. The norm is always positive.

$$\text{Norm } |\alpha\rangle = \sqrt{(|\alpha\rangle, |\alpha\rangle)}$$

If:

$$\text{Norm } |\alpha\rangle = 1$$

then the vector is called **normalized**.

4) Hilbert Space

If a linear vector space also satisfies the following three conditions, it is called a **Hilbert space**:

1. The scalar product must be defined.
2. The space must be infinite-dimensional (this means having infinitely many linearly independent ket vectors).
3. The norms of these infinite kets must be finite.

These conditions are quite restrictive. Instead, a more practical concept in mathematics is the definition of the **Rigged Hilbert Space**, which has slightly relaxed conditions.

Rigged Hilbert Space

1. The scalar product is defined.
2. The space can be either finite or infinite-dimensional.
3. The norms can be either finite or *limited infinite*.

What does "limited infinite" mean? It means something that appears infinite to us but is not truly infinite in reality.

Dirac Space (D)

We impose an additional fourth condition on the **Rigged Hilbert Space**.

Conditions

1. **Linearity I:** In the scalar product, the second factor must be linear (must satisfy linearity under scalar multiplication).

$$(|\alpha\rangle, c|\beta\rangle) = c(|\alpha\rangle, |\beta\rangle)$$

2. **Linearity II:** The second factor, under ket vector addition, must satisfy linearity.

$$(|\alpha\rangle, |\beta\rangle + |\gamma\rangle) = (|\alpha\rangle, |\beta\rangle) + (|\alpha\rangle, |\gamma\rangle)$$

3. **Antilinearity in the First Factor:**

$$(|\alpha\rangle, |\beta\rangle) = (|\beta\rangle, |\alpha\rangle)^*$$

Swapping the terms in the scalar product introduces a complex conjugation.

If we rewrite the scalar product by swapping terms:

$$(c|\alpha\rangle, |\beta\rangle) = (|\beta\rangle, c|\alpha\rangle)^*$$

$$= [(|\beta\rangle, |\alpha\rangle)c]^*$$

$$= c^*(|\beta\rangle, |\alpha\rangle)$$

$$= c^*(|\alpha\rangle, |\beta\rangle)$$

This demonstrates that the first term satisfies **antilinearity** under complex conjugation.

Notes

- c^* represents the **complex conjugate**.
- $\dagger$ represents the **Hermitian conjugate**.

iv) Norm Definition

The norm of a vector is defined as:

$$\text{Norm}[|\alpha\rangle] = \sqrt{\langle\alpha|\alpha\rangle}$$

Dirac Notation: Scalar Product

$$(|\alpha\rangle, |\beta\rangle) \equiv \langle\alpha|\beta\rangle$$

$$\text{Norm}[|\alpha\rangle] = \sqrt{\langle\alpha|\alpha\rangle}$$

Note:

- The notation $|$ and $\langle$ are used instead of parentheses for clarity in Dirac notation.

Final Form of Our Definition

With this notation, our definition takes the following form:

- (i) $\langle\alpha|(c|\beta\rangle) = c\langle\alpha|\beta\rangle$
- (ii) $\langle\alpha|(|\beta\rangle + |\gamma\rangle) = \langle\alpha|\beta\rangle + \langle\alpha|\gamma\rangle$
- (iii) $\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*$
- (iv) $\text{Norm}[|\alpha\rangle] = \langle\alpha|\alpha\rangle$

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II. Linear Operators

1. Definition

A linear operator (L.O.) is denoted as:

$$L.O. = A, B, \dots, Z$$

A linear operator transforms one ket vector into another ket vector.
There are three fundamental rules:

1. Closure:

$$|\alpha\rangle \in D \quad \Rightarrow \quad A|\alpha\rangle \in D$$

2. Linearity I:

$$A(c|\alpha\rangle) = c(A|\alpha\rangle)$$

3. Linearity II:

$$A(|\alpha\rangle + |\beta\rangle) = A|\alpha\rangle + A|\beta\rangle$$

Note: Matrices are linear operators. More precisely, matrices provide a representation for linear operators.

2. Operator Products and Related Concepts

If:

$$B|\alpha\rangle = |\beta\rangle$$

$$A|\beta\rangle = |\omega\rangle$$

then:

$$AB|\alpha\rangle = |\omega\rangle$$

This means that the operator AB transforms $|\alpha\rangle$ into $|\omega\rangle$.

Related Concepts

(i) Identity Operator:

$$I|\alpha\rangle = |\alpha\rangle$$

The identity operator is the unit operator that maps a vector to itself.

(ii) Powers of an Operator:

$$A^N$$

If $A = B$, then:

$$AB = A^2, \quad AB^2 = A^3, \quad \dots$$

(iii) Inverse Operators and Function of an Operator:

$$A^{-1}, \quad A^{-N}, \quad F(A)$$

These can be defined in certain conditions.

Operators in Dirac Space

In Dirac space:

- The operator B transforms $|\alpha\rangle$ into $|\beta\rangle$. - The inverse operator B^{-1} transforms $|\beta\rangle$ back into $|\alpha\rangle$.

$$B^{-1}B|\alpha\rangle = |\alpha\rangle$$

$$|\alpha\rangle = B^{-1}|\beta\rangle$$

Commutator Algebra

(iv) Non-commutativity:

$$AB \neq BA$$

This means that A and B do not commute, i.e., their order matters.

1. Definition of the Commutator:

$$[A, B] = AB - BA$$

This expression is called the **commutator**.

2. If:

$$[A, B] = 0$$

then A and B are **commutative** (they can be interchanged).

3. If:

$$[A, B] \neq 0$$

then A and B are **non-commutative** (their order affects the result).

Commutator Properties

1. **Antisymmetry:**

$$[A, B] = -[B, A]$$

2. **Commutator with itself:**

$$[A, A] = 0$$

3. **Commutator with any function of A :**

$$[A, F(A)] = 0$$

where $F(A)$ is any function of A .

Expanding the function $F(A)$:

$$F(A) = \sum_n c_n A^n$$

Taking the commutator:

$$\begin{aligned} [A, F(A)] &= [A, \sum_n c_n A^n] \\ &= \sum_n c_n [A, A^n] \\ &= \sum_n c_n (AA^n - A^n A) = 0 \end{aligned}$$

which proves the property.

Additional Commutator Properties

- If c is a scalar:

$$[A, c] = 0$$

- Commutator with a sum:

$$[A, (B + C)] = [A, B] + [A, C]$$

- Commutator with a product (Leibniz Rule):

$$[A, BC] = [A, B]C + B[A, C]$$

Proof:

$$[A, BC] = ABC - BCA$$

Expanding:

$$B[A, C] = BAC - BCA$$

$$[A, B]C = ABC - BAC$$

Summing both terms:

$$[A, BC] = [A, B]C + B[A, C]$$

Jacobi Identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

Proof:

$$[A, (BC - CB)] + [B, (CA - AC)] + [C, (AB - BA)] = 0$$

Expanding:

$$A(BC - CB) - (BC - CB)A + B(CA - AC) - (CA - AC)B + C(AB - BA) - (AB - BA)C$$

Rearranging terms:

$$(ABC - ACB - BCA + CBA) + (BCA - CBA - CAB + ACB) + (CAB - ACB - ABC + BAC) = 0$$

$$0 = 0 \quad \checkmark$$

This proves the Jacobi identity.

3) Groups (G)

$$G = \{g_1, g_2, \dots, g_N\}$$

where $g_1, g_2, \dots, g_N$ represent elements of the group (such as linear operators).

For a set G to be a group, it must satisfy the following four conditions:

1. Closure Condition:

If $g_i \in G$ and $g_j \in G$, then their product must also belong to G :

$$g_i g_j \in G$$

2. Identity Element:

There must exist an identity element I in G such that:

$$I \in G$$

3. Inverse Element:

For every $g_i \in G$, there must be an inverse element g_i^{-1} such that:

$$g_i^{-1} \in G$$

4. Associativity Property:

The elements must satisfy the associativity rule:

$$(g_i g_j) g_k = g_i (g_j g_k)$$

If these four conditions are met, the set G is a group. Otherwise, we cannot call it a group.

Order of a Group

The order of a group N can be:

$$N = \begin{cases} \text{Finite} \\ \text{limited Infinite} \\ \text{truly infinite} \end{cases}$$

In the case of an uncountable group, the group elements can be expressed as functions depending on a continuous parameter.

Abelian Groups

If a group is commutative, it is called an **Abelian group**. This means that:

$$g_i g_j = g_j g_i$$

which implies:

$$[g_i, g_j] = 0$$

Lie Groups

Lie groups are **continuous groups**, meaning that their elements depend on one or more continuous parameters.

If all parameters are set to zero, the identity element of the group must be obtained in a Lie Group:

$$g_\alpha = g(\alpha), \quad g(0) = \mathbf{1}$$

Example:

$$g(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$$

This is an example of an **Abelian group** and Lie Group (its parameters are continuous and gives the identity matrix for $\alpha=0$).

$$g(\alpha)g(\beta) = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix}$$

Multiplying the matrices:

$$= \begin{pmatrix} \cos \alpha \cos \beta - \sin \alpha \sin \beta & -\cos \alpha \sin \beta - \sin \alpha \cos \beta \\ \sin \alpha \cos \beta + \cos \alpha \sin \beta & \cos \alpha \cos \beta - \sin \alpha \sin \beta \end{pmatrix}$$

Using trigonometric identities:

$$= \begin{pmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{pmatrix} \\ = g(\beta)g(\alpha)$$

Since:

$$g(\alpha)g(\beta) = g(\beta)g(\alpha)$$

the group is **commutative**, meaning it is an **Abelian Lie group**.

SO(2) Group

This group is an **SO(2) group**, where:

- **SO** stands for **Special Orthogonal**. - **Special** means that the determinant is equal to 1:

$$\det g(\alpha) = 1$$

Groups with determinant 1 are called **special groups (S)**.

- **Orthogonal** means that different group elements are orthogonal matrices.

- The group is **two-dimensional**.

The SO(2) group is a **rotation group**.

A group whose elements transform a vector from one coordinate system to another via rotation is called a **rotation group**.

The fact that SO(2) is Abelian means that rotating a vector to the right by an angle α or to the left by the same angle makes no difference. The only difference is in the coordinate system.

$$SO(2) \rightarrow \text{Lie Group} \rightarrow \text{Abelian Lie Group}$$

4) Similarity Transformation

$$A|\alpha\rangle = |\beta\rangle$$

Here, the operator A transforms $|\alpha\rangle$ into $|\beta\rangle$.

Now, consider another transformation. Let the operator S transform $|\beta\rangle$ into $|\beta'\rangle$, meaning:

$$S|\beta\rangle = |\beta'\rangle$$

Applying S to both sides of $A|\alpha\rangle = |\beta\rangle$:

$$SA|\alpha\rangle = S|\beta\rangle$$

Using the given relation:

$$SAS^{-1}S|\alpha\rangle = S|\beta\rangle$$

Since $SS^{-1} = I$, this simplifies to:

$$A'|\alpha'\rangle = |\beta'\rangle$$

where:

$$A' = SAS^{-1}$$

Thus, kets transform as:

$$|\alpha\rangle \rightarrow |\alpha'\rangle, \quad |\beta\rangle \rightarrow |\beta'\rangle$$

whereas operators transform under ****similarity transformation**** as:

$$A' = SAS^{-1}$$

This means we are performing a ****reference frame transformation**** from one coordinate system to another.

This is essentially a type of coordinate system transformation:

$$|\alpha\rangle \rightarrow |\alpha'\rangle, \quad |\beta\rangle \rightarrow |\beta'\rangle, \quad A \rightarrow A'$$

That is, we are transitioning a physical system from one coordinate system to another.

Physical laws should be independent of the coordinate system. This means they must remain invariant under transformations.

$$A'|\alpha'\rangle = |\beta'\rangle$$

Here, the **form remains unchanged**, but the coordinates transform.

5) Matrix Elements

The matrix element:

$$\langle\alpha|A|\beta\rangle$$

is a complex (or real) number. In general complex number.

$$\langle\alpha|A|\beta\rangle$$

is a complex number representing the matrix elements of operator A .

Each matrix element represents the value of A at the row α and column β :

$$A = (a_{\alpha\beta})$$

We define linear operators as operators that act on ket vectors. Linear operators always act on ket vectors, does not act on bra vectors.

To understand a linear operator (L.O.), we need to know its matrix elements.

6) Hermitian Conjugate

Hermitian conjugation is defined in terms of the matrix elements of an operator:

$$A \rightarrow A^\dagger$$

Matrix Elements

Examining the matrix elements:

$$\langle \alpha | A^\dagger | \beta \rangle = \langle \beta | A | \alpha \rangle^*$$

- The **row index** (α) and **column index** (β) are swapped. - The complex conjugate is taken. - This means **transpose + complex conjugation**.

Properties

(i) **Double Conjugation:**

$$(A^\dagger)^\dagger = A$$

(ii) **Sum Rule:**

$$(A + B)^\dagger = A^\dagger + B^\dagger$$

(iii) **Scalar Multiplication:**

$$(cA)^\dagger = c^* A^\dagger$$

where c is a complex number.

(iv) **Product Rule:**

$$(AB)^\dagger = B^\dagger A^\dagger$$

7) Hermitian and Unitary Linear Operators

Hermitian Operators

$$\langle \alpha | H | \beta \rangle = \langle \alpha | H^\dagger | \beta \rangle$$

This implies:

$$H = H^\dagger$$

Thus, H is a **Hermitian operator**.

Unitary Operators

$$\langle \alpha | U^\dagger | \beta \rangle = \langle \alpha | U^{-1} | \beta \rangle$$

which leads to:

$$U^{-1} = U^\dagger$$

Thus, U is a **unitary operator**.

Relation Between Hermitian and Unitary Operators

There is a strong connection between these two types of operators. Every unitary operator can be written as:

$$U = e^{iH}$$

Let us show :

$$U^{-1} = e^{-iH}$$

$$U^\dagger = e^{-iH^\dagger}$$

Since we know:

$$U^{-1} = U^\dagger$$

it follows that:

$$e^{-iH} = e^{-iH^\dagger}$$

which implies:

$$H = H^\dagger$$

Thus, **H** must be Hermitian.

Quantum Mechanics Interpretation

In **Quantum Mechanics**:

- **Observables** correspond to **linear operators** (L.O.). - **States** correspond to **ket vectors**.